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On the unified approach to anisotropic and isotropic elasticity for singularity, interface and crack in dissimilar media

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Abstract

Proposed in this paper is the equivalence between anisotropic and isotropic elasticity for two-dimensional deformation under certain conditions. That is, the isotropic elasticity can be reconstructed in the same framework of the anisotropic elasticity, when the interface between dissimilar media lies along a straight line. Therefore, many known solutions for an anisotropic bimaterial are valid for a bimaterial, of which one or both of the constituent materials are isotropic. The usefulness of the equivalence is that the solutions for singularities and cracks in an anisotropic/isotropic bimaterial can easily be obtained without solving the boundary value problems directly. The interaction solutions of singularities, interfaces, and cracks in infinite anisotropic bimaterial are summarized, to be used for the cases of isotropic/isotropic and anisotropic/isotropic bimaterials. Conservation integrals also have the similar analogy between anisotropic and isotropic elasticity so that J integral and J-based mutual integral $\mathscr{M}$ are expressed in the same complex forms for anisotropic and isotropic materials, when both end points of the integration paths are on the straight interface. The use of J and $\mathscr{M}$ integrals together with the present equivalence are exemplified to obtain energy release rate, stress intensity factors, and T-stresses of interfacial cracks lying in the interface of anisotropic/anisotropic, isotropic/isotropic, or anisotropic/isotropic solids.

Keywords: Anisotropic elasticity; Isotropic elasticity; Singularity; Crack; Bimaterial; Conservation integral

1. Introduction

In 2-D linear elasticity, an interesting question often arises: how can a solution for isotropic material be obtained from a solution for anisotropic material with the same geometry and boundary condition? For trivial case in which the in-plane stresses are prescribed at infinity the solution for the stress is independent of the elastic constant, hence, the relation between the solutions for the isotropic and anisotropic material is

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obvious; they are the same. If, on the other hand a single dislocation lies in infinite isotropic medium, its solution for stresses is different from that for anisotropic medium with the otherwise identical conditions.

Since the isotropy is a very special case of anisotropy, it is no surprise that some of the isotropic solutions can be obtained from the anisotropic solutions by simply using the isotropic constants. The difficulties encountered in obtaining the isotropic solution from its anisotropic counterpart are more of an artifact of the formulation used in obtaining the anisotropic elasticity solution. Two-dimensional isotropic elasticity is usually formulated by the potentials of the elegant works of Muskhelishvili (1953), which are the stepstones for various kinds of 2-D isotropy problems. On the other hand, the most commonly used formulation in anisotropic elasticity is from Lekhnitskii (1963); Eshelby et al. (1953) and Stroh (1958), known as Stroh formalism that represents the displacements and stresses in terms of a complex function vector $\mathbf{f}(z)$ and two complex matrices, $\mathbf{A}$ and $\mathbf{B}$ (these quantities will be defined later in this study). One would expect that Muskhelishvili's complex potentials could be easily obtained from the corresponding complex function vector $\mathbf{f}(z)$ when the material is isotropic. But the relation has not been clearly investigated. In calculating $\mathbf{A}$ and $\mathbf{B}$, an eigenvalue problem must be solved. It turns out that the corresponding eigenvalue problem has repeated eigenvalues when the material is isotropic. This creates difficulties when constructing $\mathbf{A}$ and $\mathbf{B}$, for they span essentially the eigenvector space that is not complete for isotropic materials.

Ting and Hwu (1988) presented a method how to construct $\mathbf{A}$ and $\mathbf{B}$ in such degenerate cases. Their method is rigorous, however, it yields the quite complicated result, and moreover does not show the relation of the anisotropic solution with the familiar potentials of Muskhelishvili (1953) in isotropic elasticity. If an anisotropic solution is written in terms of $\mathbf{A}$ and $\mathbf{B}$, the corresponding isotropic solution cannot be obtained by simply using the isotropic elasticity constants. However, if an anisotropic solution can be written in certain combinations of $\mathbf{A}$ and $\mathbf{B}$, so that the solution is expressed as a function of the Barnett–Lothe tensors (Barnett and Lothe, 1973), then the issue of repeated eigenvalues can be circumvented. The Barnett–Lothe tensors can be computed from the elastic constants through integration without the need of solving the eigenvalue problem. In this case, the isotropic solution is obtained from the anisotropic solution by simply using the isotropic elastic constants. Such reductions from anisotropic to isotropic solutions have been discussed in Qu and Li (1991) and Qu et al. (1992) for interfacial dislocations, and in Qu and Bassani (1993) for interfacial cracks. Of course, not all bimaterial solutions can be expressed in terms of the Barnett–Lothe tensors, an example being the Green's function in a bimaterial (Qu, 1992). In case like this, the isotropic solutions cannot be obtained by simply using the isotropic elastic constants in the anisotropic solutions.

One of the aims of the present work is to put the anisotropic and isotropic formulations in as similar forms as possible, and consequently the simple and straightforward contrast between the analytic functions in Stroh formalism for anisotropic elasticity and the potentials by Muskhelishvili for isotropic elasticity are presented. As a result, the explanation regarding the inaccessibility to isotropy solution from the more general anisotropic solution is elaborated on, and a method to guess the isotropic solution from the anisotropic solution for the cases of a singularity (such as a point force and dislocation) and a uniform stress applied at infinity is suggested and some comparisons of the structures of the solutions are made.

As the bimaterial is in wide use, the solutions for the bimaterial system became important, and Suo (1989) presented the solution for isotropic bimaterial. The same author (Suo, 1990) also solved the anisotropic bimaterial problem. The former work is formulated in terms of the Muskhelishvili potential while the latter work in terms of the analytic functions based on the Stroh formalism. Both works make use of the analytic continuation argument in order to get the solutions for the cases of a point singularity, or the surface tractions on the interfacial crack surface. It is found in this work that the solution for isotropy is easily obtained from that for anisotropy, and vice versa. Also the present method is applicable for anisotropic/isotropic bimaterial system, which is now used in MEMS packaging of glass/silicon bonding (Go and Cho, 1999; Labossiere et al., 2002).

The J integral (Rice, 1968) has been extensively studied for crack in homogeneous materials as well as in bimaterials. For isotropic material this integral was written in terms of the Muskhelishvili potential

(Budiansky and Rice, 1973) while for anisotropic material Yeh et al. (1993) and recently Kim et al. (2001) obtained the J integral in terms of the analytic functions based on the Stroh formalism. It is now found that as long as the crack lies along the interface, both cases are written in the same form, showing the advantage of the present approach. Closely related to this topic is the mutual integral $\mathscr{M}$ proposed by Chen and Shield (1977), which is path-independent and has been used together with the J integral for separation of Mode I, II and III (Cho et al., 1994). This $\mathscr{M}$ integral is also shown to be written in the same form regardless of whether the solids comprise of isotropic (homogeneous), anisotropic (homogeneous), isotropic/isotropic, anisotropic/anisotropic or isotropic/anisotropic materials.

2. Linear theory of elasticity for 2D deformation

2.1. Anisotropic elasticity

We begin with the brief review of anisotropic elasticity by considering a generalized two-dimensional deformation, in which the displacements u_j depend only on x_1 and x_2. The constitutive equations for a linear elastic material are

$$\sigma_{ij} = \sum_{k,m=1}^{3} C_{ijkm} \frac{\partial u_k}{\partial x_m}, \quad (i,j = 1,2,3) \tag{2.1}$$

in which σ_{ij} are the stresses and C_{ijkm} the elastic constants. The equations of equilibrium are

$$\sum_{j,k,m=1}^{3} C_{ijkm} \frac{\partial^2 u_k}{\partial x_j \partial x_m} = 0. \tag{2.2}$$

A general solution for the displacements satisfying Eq. (2.2) and the corresponding stresses may be written as (Eshelby et al., 1953; Stroh, 1958)

$$\left\{ \frac{\partial u_i}{\partial x_1} \right\} = 2\mathrm{Re}[\mathbf{A}\mathbf{f}'(z)], \tag{2.3a}$$

$$\{\sigma_{2i}\} = 2\mathrm{Re}[\mathbf{B}\mathbf{f}'(z)], \tag{2.3b}$$

$$\{\sigma_{1i}\} = 2\mathrm{Re}[\mathbf{\Lambda}\mathbf{f}'(z)], \tag{2.3c}$$

where

$$\mathbf{f}'(z) = \left[f_1'(z_1), f_2'(z_2), f_3'(z_3)\right]^{\mathrm{T}}. \tag{2.4}$$

Eq. (2.3a) is the differentiated form of

$$\{u_i\} = 2\mathrm{Re}[\mathbf{A}\mathbf{f}(z)]. \tag{2.5}$$

Here and throughout the paper, the overbar represents the complex conjugate and the prime, the derivative with respect to the associated argument. The functions $f_j(z_j)$ are analytic functions of complex variable $z_j = x_1 + p_j x_2$. Each column of $\mathbf{A}$ and each of p_j's are the eigenvector and the eigenvalue with positive imaginary part, respectively, of the sextic equation

$$\sum_{k=1}^{3} \left[C_{i1k1} + p_\alpha(C_{i1k2} + C_{i2k1}) + p_\alpha^2 C_{i2k2}\right] A_{k\alpha} = 0. \tag{2.6}$$

The matrix **B** is given by

$$B_{ij} = \sum_{k=1}^{3}(C_{i2k1} + p_j C_{i2k2})A_{kj}, \tag{2.7}$$

and

$$\Lambda_{ij} = -B_{ij}p_j \quad (\text{not sum over } j). \tag{2.8}$$

Throughout this paper we do not employ summation rule unless stated otherwise. The matrices **A** and **B** are not unique in the sense that any arbitrary constant can be multiplied to the eigenvectors (the column vectors of **A** and **B**). To get the similarity between anisotropic and isotropic elasticity in the subsequent sections, the arbitrary normalizing factor is chosen so that the matrix **B** is non-dimensional, and the matrix **A** has the dimension of compliance. If Eq. (2.6) has three distinct pairs of complex roots on which we are concentrating (the isotropic solid having multiple roots of Eq. (2.6) is dealt with in the subsequent subsection), the matrices **A** and **B** are non-singular and may be used to define

$$\mathbf{M}^{-1} \equiv \mathrm{i}\mathbf{A}\mathbf{B}^{-1}, \tag{2.9}$$

which is a positive definite Hermitian matrix (Stroh, 1958). Here, $\mathrm{i} = \sqrt{-1}$ and $(\ )^{-1}$ stands for the inverse of the matrix. Explicit expressions of **A**, **B**, and **M** in terms of elastic constants are given in Suo (1990) and Ting (1996) and we present them for cubic materials in Appendix A for future use in examples.

Of particular importance are the following derivatives of displacements and tractions that must be continuous across the perfectly-bonded interface $x_2 = 0$:

$$\left\{\frac{\partial u_i}{\partial x_1}(x_1)\right\} = \mathbf{A}\mathbf{f}'(x_1) + \bar{\mathbf{A}}\bar{\mathbf{f}}'(x_1), \tag{2.10a}$$

$$\{\sigma_{2i}(x_1)\} = \mathbf{B}\mathbf{f}'(x_1) + \bar{\mathbf{B}}\bar{\mathbf{f}}'(x_1), \tag{2.10b}$$

$$\{\sigma_{1i}(x_1)\} = \mathbf{\Lambda}\mathbf{f}'(x_1) + \bar{\mathbf{\Lambda}}\bar{\mathbf{f}}'(x_1). \tag{2.10c}$$

Eq. (2.10a) is equivalent to the continuity of displacements. A function $f(z)$ is an analytic function of $z = x_1 + px_2$ for $x_2 > 0$ (or $x_2 < 0$) for any p if it is analytic for $x_2 > 0$ (or $x_2 < 0$) for one p, where p is any complex number with positive imaginary part (Suo, 1990), therefore, one can refer to $f_j(z)$ instead of $f_j(z_j)$ and, if necessary, z is reinterpreted as z_j.

2.2. Isotropic elasticity

The components of the stresses and displacements for an isotropic body under plane deformation are expressed in terms of Muskhelishvili complex potentials for inplane and antiplane problems as follows (Muskhelishvili, 1953; Cho et al., 1994):

$$\sigma_{11} + \sigma_{22} = 2[\Phi(z) + \overline{\Phi(z)}], \tag{2.11a}$$

$$\sigma_{22} + \mathrm{i}\sigma_{12} = \overline{\Phi(z)} + \Omega(z) + (\bar{z} - z)\Phi'(z), \tag{2.11b}$$

$$\sigma_{32} + \mathrm{i}\sigma_{31} = \omega(z), \tag{2.11c}$$

$$-2\mathrm{i}G\frac{\partial}{\partial x_1}(u_2 + \mathrm{i}u_1) = \kappa\overline{\Phi(z)} - \Omega(z) - (\bar{z} - z)\Phi'(z), \tag{2.11d}$$

$$\frac{\partial u_3}{\partial x_1} = \frac{1}{2G\mathrm{i}}\left[\omega(z) - \overline{\omega(z)}\right], \tag{2.11e}$$

where $\kappa = 3 - 4v$ for plane strain and $(3 - v)/(1 + v)$ for plane stress, v and G are Poisson's ratio and shear modulus, respectively. It is known that the isotropic elasticity is a special case of the anisotropic elasticity, i.e., when the three eigenvalues of Eq. (2.6) degenerate and $p_1 = p_2 = p_3 = \mathrm{i}$, and furthermore, only two independent eigenvectors can be found (Ting, 1996). With a view to relate the potentials $\Phi(z)$, $\Omega(z)$ and $\omega(z)$ in Eqs. (2.11a)–(2.11e) with $\mathbf{f}'(z)$ in Eqs. (2.3a)–(2.3c) and Eq. (2.4), we rewrite Eqs. (2.11a)–(2.11e) as follows:

$$\left\{\frac{\partial u_i}{\partial x_1}\right\} = 2\mathrm{Re}[\mathbf{A}^*\mathbf{g}'(z)], \tag{2.12a}$$

$$\{\sigma_{2i}\} = 2\mathrm{Re}[\mathbf{B}^*\mathbf{g}'(z)], \tag{2.12b}$$

$$\{\sigma_{1i}\} = 2\mathrm{Re}[\mathbf{\Lambda}^*\mathbf{g}'(z)], \tag{2.12c}$$

where

$$\mathbf{g}'(z) = [\Phi(z), \Omega(z) + (\bar{z} - z)\Phi'(z), \omega(z)]^{\mathrm{T}} = \mathbf{f}'^*(z) + (\bar{z} - z)\mathbf{Q}^* \cdot \mathbf{f}''^*(z), \tag{2.13a}$$

$$\mathbf{f}'^*(z) = [\Phi(z), \Omega(z), \omega(z)]^{\mathrm{T}}, \tag{2.13b}$$

$$\mathbf{A}^* = \frac{1}{4G\mathrm{i}}\begin{bmatrix} \kappa\mathrm{i} & -\mathrm{i} & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{2.14a}$$

$$\mathbf{B}^* = \frac{1}{2}\begin{bmatrix} \mathrm{i} & -\mathrm{i} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.14b}$$

$$\mathbf{\Lambda}^* = \frac{1}{2}\begin{bmatrix} 3 & -1 & 0 \\ \mathrm{i} & -\mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \end{bmatrix}, \tag{2.14c}$$

$$\mathbf{Q}^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.14d}$$

Take Eq. (2.12b) as an example for proof. From Eqs. (2.11b) and (2.11c),

$$\sigma_{21} = \frac{1}{2\mathrm{i}}\left[\overline{\Phi(z)} + \Omega(z) + (\bar{z} - z)\Phi'(z) - \Phi(z) - \overline{\Omega(z)} - (z - \bar{z})\overline{\Phi'(z)}\right], \tag{2.15a}$$

$$\sigma_{22} = \frac{1}{2}\left[\overline{\Phi(z)} + \Omega(z) + (\bar{z} - z)\Phi'(z) + \Phi(z) + \overline{\Omega(z)} + (z - \bar{z})\overline{\Phi'(z)}\right], \tag{2.15b}$$

$$\sigma_{23} = \frac{1}{2}\left[\omega(z) + \overline{\omega(z)}\right], \tag{2.15c}$$

hence,

$$\{\sigma_{2i}\} = \mathbf{B}^*\mathbf{f}'^*(z) + \bar{\mathbf{B}}^*\bar{\mathbf{f}}'^*(\bar{z}) + (\bar{z} - z)\Phi'(z)\mathbf{b}^* + (\overline{\bar{z} - z})\overline{\Phi'(z)}\bar{\mathbf{b}}^*, \tag{2.16}$$

where $\mathbf{f}'^{*}(z)$ is defined in Eq. (2.13b) and $\mathbf{b}^{*} = [-\mathrm{i}/2, 1/2, 0]^{\mathrm{T}}$. Eq. (2.16) is further rewritten by introducing $\mathbf{g}'(z)$ and $\mathbf{Q}^{*}$ (Eqs. (2.13a) and (2.14d)), yielding Eq. (2.12b). In a similar way, Eqs. (2.12a) and (2.12c) can be justified.

Several points are due: (i) Notice the similarity between Eqs. (2.3a) and (2.12a), Eqs. (2.3b) and (2.12b), Eqs. (2.3c) and (2.12c), and Eqs. (2.4) and (2.13b), respectively, and the difference between the isotropic and anisotropic elasticity as is evident from Eq. (2.13a). This difference makes the two different approaches justifiable, taken by Muskhelishvili (1953) and Stroh (1958), respectively. (ii) When $x_2 = 0$,

$$\mathbf{g}'(x_1) = \mathbf{f}'^{*}(x_1) = [\Phi(x_1), \Omega(x_1), \omega(x_1)]^{\mathrm{T}}, \tag{2.17a}$$

$$\mathbf{f}'(x_1) = [f_1'(x_1), f_2'(x_1), f_3'(x_1)]^{\mathrm{T}}, \tag{2.17b}$$

and consequently, the difference between the Muskhelishvili potentials and the Stroh analytic functions disappears if one identifies $\mathbf{f}'^{*}(x_1)$ as $\mathbf{f}'(x_1)$. As will be seen later, this observation has significant consequences when we are dealing with the bimaterial whose interface is along the x_1-axis (and even with an interfacial crack along the x_1-axis).

We, therefore, write the derivative of the displacements and stresses when $x_2 = 0$ as

$$\left\{ \frac{\partial u_i}{\partial x_1}(x_1) \right\} = \mathbf{A}^{*}\mathbf{f}'^{*}(x_1) + \bar{\mathbf{A}}^{*}\bar{\mathbf{f}}'^{*}(x_1), \tag{2.18a}$$

$$\{\sigma_{2i}(x_1)\} = \mathbf{B}^{*}\mathbf{f}'^{*}(x_1) + \bar{\mathbf{B}}^{*}\bar{\mathbf{f}}'^{*}(x_1), \tag{2.18b}$$

$$\{\sigma_{1i}(x_1)\} = \mathbf{\Lambda}^{*}\mathbf{f}'^{*}(x_1) + \bar{\mathbf{\Lambda}}^{*}\bar{\mathbf{f}}'^{*}(x_1). \tag{2.18c}$$

The positive definite Hermitian matrix $\mathbf{M}^{*}$, which will be useful in the following sections, is defined by

$$\mathbf{M}^{*-1} \equiv \mathrm{i}\mathbf{A}^{*}\mathbf{B}^{*-1} = \frac{1}{4G}\begin{bmatrix} \kappa+1 & (\kappa-1)\mathrm{i} & 0 \\ -(\kappa-1)\mathrm{i} & \kappa+1 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \tag{2.19}$$

2.3. Equivalence between anisotropic and isotropic elasticity

The reformulation of isotropic elasticity described in the previous Section 2.2 gives us some analogy between anisotropic and isotropic elasticity. That is, referring to Eqs. (2.10a)–(2.10c) and (2.18a)–(2.18c), the continuity conditions of the derivatives of displacements and tractions along x_1-axis are expressed in the identical mathematical forms. Therefore, the matrices $\mathbf{A}^{*}$, $\mathbf{B}^{*}$, $\mathbf{\Lambda}^{*}$, and $\mathbf{f}^{*}(z)$ for an isotropic material correspond to $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Lambda}$, and $\mathbf{f}(z)$ for an anisotropic material, respectively. This analogy results in the following equivalence theorem.

Equivalence theorem. *In employing the analytic continuation during the solution procedure for a bimaterial undergoing two dimensional deformation, the Stroh formalism of the anisotropic elasticity can be used regardless of whether the bimaterial is comprised of anisotropic/anisotropic, isotropic/isotropic, or isotropic/anisotropic materials, provided that the interface of the bimaterial is along a straight line, say, x_1-axis.*

Hereafter, we do not distinguish $\mathbf{A}^{*}$, $\mathbf{B}^{*}$, $\mathbf{\Lambda}^{*}$, $\mathbf{M}^{*}$, and $\mathbf{f}^{*}(z)$ from $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Lambda}$, $\mathbf{M}$, and $\mathbf{f}(z)$, respectively, unless stated otherwise. The equivalence theorem proposed in this paper provides us with the solutions for isotropic solids transcribable from those for anisotropic solids as will be seen later. Even though the interface is partially debonded, that is, there exists an interfacial crack, the equivalence is valid, which is exploited in Section 3.2. Furthermore, conservation integrals also have the same analogy so that J integral and J-based mutual integral are expressed in the same complex forms for anisotropic and isotropic materials, when both end points of the integration paths are on the straight interface (Section 4). These conservation integrals are applied to obtain the energy release rate, the stress intensity factors, and T-stresses of interfacial cracks (Section 4). The above equivalence between anisotropic and isotropic elasticity is also valid for homogeneous materials and semi-infinite bodies with free or fixed surface, which are the special cases of a bimaterial.

2.4. Generalized Dundurs parameter

Generalized Dundurs parameters for dissimilar anisotropic materials are as follows (Beom and Atluri, 1995; Ting, 1995):

$$\boldsymbol{\alpha} = (\mathbf{L}_1 - \mathbf{L}_2)(\mathbf{L}_1 + \mathbf{L}_2)^{-1}, \quad \boldsymbol{\beta} = (\mathbf{L}_1^{-1} + \mathbf{L}_2^{-1})^{-1}(\mathbf{W}_1 - \mathbf{W}_2), \tag{2.20}$$

in which the subscripts 1 and 2 stand for materials 1 and 2, respectively, and

$$\mathbf{L}^{-1} = \mathrm{Re}\{\mathbf{M}^{-1}\} \tag{2.21}$$

is a symmetric real matrix and

$$\mathbf{W} = -\mathrm{Im}\{\mathbf{M}^{-1}\} \tag{2.22}$$

is an antisymmetric real matrix. It is noted that the two real matrices **L** and **W** have smooth limits even if **A** and **B** become singular. If a final result involves only the matrices **L** and **W** (also $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ but not **A** and **B**), it is also valid for any degenerate cases. The real matrices **L** and **W** can be calculated directly from the elastic constants without solving the eigenvalue problem, such as the Barnett–Lothe tensors (Barnett and Lothe, 1973). The stress fields in a bimaterial, with tractions prescribed on its outer boundary, depend only on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, regardless of whether the constituent materials of the bimaterial are anisotropic/anisotropic, isotropic/isotropic or anisotropic/isotropic (Beom and Atluri, 1995; Ting, 1995). Another bimaterial parameter ε, the oscillatory index, is related to $\boldsymbol{\beta}$ by

$$\varepsilon = \frac{1}{2\pi}\ln\left(\frac{1-\beta}{1+\beta}\right), \quad \beta = \left[-(1/2)\mathrm{tr}(\boldsymbol{\beta}^2)\right]^{1/2}, \tag{2.23}$$

where $\mathrm{tr}(\cdot)$ means the trace of $(\cdot)$. An interfacial crack between dissimilar materials 1 and 2 has non-oscillatory character if $\varepsilon = 0$. For the isotropic/isotropic bimaterial case, the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as computed from Eqs. (2.20)–(2.22) together with Eq. (2.19) reduce to

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.24}$$

in which α, β, and γ, respectively, are two Dundurs parameters for inplane and one for antiplane deformation of an isotropic solid defined as (Dundurs, 1969)

$$\alpha = \frac{G_1(\kappa_2+1) - G_2(\kappa_1+1)}{G_1(\kappa_2+1) + G_2(\kappa_1+1)}, \quad \beta = \frac{G_1(\kappa_2-1) - G_2(\kappa_1-1)}{G_1(\kappa_2+1) + G_2(\kappa_1+1)}, \quad \gamma = \frac{G_1 - G_2}{G_1 + G_2}. \tag{2.25}$$

For the isotropic/cubic bimaterial case, as seen in Appendix A, the generalized Dundurs parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined in Eq. (2.20) reduce to the same forms as in Eq. (2.24), in which α, β, and γ are replaced by

$$\alpha = \left(\frac{4G}{\kappa+1} - \frac{C_{11}^2 - C_{12}^2}{C_{11}\sqrt{\eta+2}}\right)\Big/\left(\frac{4G}{\kappa+1} - \frac{C_{11}^2 - C_{12}^2}{C_{11}\sqrt{\eta+2}}\right), \tag{2.26a}$$

$$\beta = \left(\frac{1}{C_{11}+C_{12}} - \frac{\kappa-1}{4G}\right)\Big/\left(\frac{\kappa+1}{4G} + \frac{C_{11}\sqrt{\eta+2}}{C_{11}^2 - C_{12}^2}\right), \tag{2.26b}$$

$$\gamma = \frac{G - C_{66}}{G + C_{66}}. \tag{2.26c}$$

Here C_{11}, C_{12}, and C_{66} are three independent elastic constants of the cubic material and η is defined in Eq. (A.2). Interestingly the stress fields in an isotropic/cubic bimaterial depend only on two Dundurs parameters α and β for inplane deformation, as can be concluded from Beom and Atluri (1995), even though the number of the independent elastic constants of the bimaterial is 5 (G, κ, C_{11}, C_{12}, and C_{66}).

3. Singularities and interfacial cracks in an anisotropic/anisotropic, isotropic/isotropic or anisotropic/isotropic bimaterial

Since the equivalence theorem described in the previous section provides us with general methods to obtain the solution for isotropic/isotropic or anisotropic/isotropic bimaterial from that for anisotropic/anisotropic bimaterial, in this section we summarize the solutions of singularities and interfacial cracks in dissimilar anisotropic media and utilize the equivalence theorem.

3.1. A singularity in a bimaterial

We take the solution for line force or dislocation at (x_1^0, x_2^0) in an infinite anisotropic, homogeneous medium as (Stroh, 1958; Suo, 1990)

$$\mathbf{f}_0'(z_j) = \left[\frac{q_1}{2\pi(z_1 - s_1)}, \frac{q_2}{2\pi(z_2 - s_2)}, \frac{q_3}{2\pi(z_3 - s_3)}\right]^{\mathrm{T}}, \quad \text{i.e.,} \quad f_{0j}'(z_j) = \frac{q_j}{2\pi(z_j - s_j)}, \tag{3.1}$$

where $s_j = x_1^0 + \mu_j x_2^0$ and $\mathbf{q} = \{q_j\}$ is related to Burgers vector $\mathbf{b}$ and force per unit length $\mathbf{p}$ as

$$\mathbf{q} = \mathbf{B}^{-1}(\mathbf{M}^{-1} + \bar{\mathbf{M}}^{-1})^{-1}\mathbf{b} - \mathbf{A}^{-1}(\mathbf{M} + \bar{\mathbf{M}})^{-1}\mathbf{p}. \tag{3.2}$$

When a singularity is located at (x_1^0, x_2^0) in an infinite isotropic, homogeneous material, the solutions $\mathbf{f}_0'(z)$ are as follows (Muskhelishvili, 1953; Suo, 1989):

$$\mathbf{f}_0'(z) = [\Phi_0(z), \Omega_0(z), \omega_0(z)]^{\mathrm{T}} = \left[-\frac{Q}{z - s}, \frac{\bar{Q}\hat{\kappa}}{z - s} - \frac{Q(\bar{s} - s)}{(z - s)^2}, \frac{q}{z - s}\right]^{\mathrm{T}}, \tag{3.3}$$

where $\hat{\kappa} = \kappa$ for a point force and $\hat{\kappa} = -1$ for a dislocation, and $s = x_1^0 + \mathrm{i}x_2^0$ is the position of the singularity. Q and q are defined in terms of the Burgers vector $\mathbf{b}$ and the force per unit length $\mathbf{p}$ as

$$Q = \frac{p_1 + \mathrm{i}p_2}{2\pi(\kappa + 1)} + \frac{G\mathrm{i}(b_1 + \mathrm{i}b_2)}{\pi(\kappa + 1)}, \quad q = \frac{p_3}{2\pi \mathrm{i}} + \frac{Gb_3}{2\pi}. \tag{3.4}$$

Eqs. (3.3) and (3.4) were obtained from the theory of isotropic elasticity using the method of Muskhelishvili, however, it is convenient to write them in a form as close as possible to Eqs. (3.1) and (3.2), namely,

$$\begin{aligned} \mathbf{f}_0'(z) = [\Phi_0(z), \Omega_0(z), \omega_0(z)]^{\mathrm{T}} &= \frac{\mathbf{q}}{2\pi(z - s)} + \frac{(\bar{s} - s)\mathbf{Q}^* \cdot \mathbf{q}}{2\pi(z - s)^2} \\ &= \left[\frac{q_1}{2\pi(z - s)}, \frac{q_2}{2\pi(z - s)} + \frac{(\bar{s} - s)q_1}{2\pi(z - s)^2}, \frac{q_3}{2\pi(z - s)}\right]^{\mathrm{T}}, \end{aligned} \tag{3.5}$$

where $\mathbf{q}$ is defined by Eq. (3.2) using isotropic $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{M}$ given in Eqs. (2.14a), (2.14b), (2.19), respectively. Note the similarity of Eq. (3.5) to Eq. (3.1). The only difference is the additional term $(\bar{s} - s)q_1/2\pi(z - s)^2$, which can be interpreted as follows: for a singularity at the origin, $f'_{02}(z) \equiv \Omega_0(z) = q_2/2\pi z$, while for a singularity at $z = s$,

$$f'_{02}(z) \equiv \Omega_0(z) = \frac{q_2}{2\pi(z-s)} + \frac{(\bar{s}-s)q_1}{2\pi(z-s)^2} \tag{3.6}$$

from the well-known result by Muskhelishvili for translation of the coordinate ($f'_{01}(z) \equiv \Phi_0(z) = q_1/2\pi z$, on the other hand, changes to $q_1/2\pi(z-s)$ (Muskhelishvili, 1953)). The difference in $f'_{02}(z)$ between the isotropic and anisotropic elasticity may be attributed to the degenerate eigenvalues of Eq. (2.6) for the case of isotropic elasticity, and this seems to be the only obstacle to be surmounted in passing from anisotropy to isotropy in elasticity (or vice versa). For the case of uniform applied stress at infinity, $f'_{01}(z) = c_1$, $f'_{02}(z) = c_2$, $f'_{03}(z) = c_3$, regardless of whether isotropy or anisotropy. Let us recall the statement in Section 1, Introduction: How can a solution for isotropic material be obtained from a solution for anisotropic material with the same geometry and boundary condition? The case of a point force is to be illustrated. Suppose the solution $\mathbf{f}'_0(z)$, Eq. (3.1) for anisotropic material with a point force at (x_1^0, x_2^0) is known. The crucial point is, "can we obtain $\mathbf{f}'_0(z)$, Eq. (3.5) for isotropic material with due care of the different nature of $f'_{02}(z)$?" As seen above, in this case, the question is no more meaningful since we have both solutions already at hand, however, it suggests some guidelines for the case of finite geometry. For the case of uniform stresses at infinity, the problem becomes trivial. The result of the equivalence theorem in bimaterial system is now explored as below.

Consider an anisotropic/anisotropic bimaterial bonded along x_1-axis as shown in Fig. 1(a), in which a singularity is located in lower half-space. Therefore, the elastic constants of material 2 are implied in the homogeneous solution $\mathbf{f}'_0(z)$. The bimaterial solution for a singularity can be constructed in terms of the homogeneous one for the same singularity by using the method of analytic continuation (Suo, 1990; Choi and Earmme, 2002a). We write the bimaterial solution of Suo (1990), though slightly different notations are used, as

$$\mathbf{f}'(z) = \begin{cases} \mathbf{B}_1^{-1}(\mathbf{I}+i\boldsymbol{\beta})^{-1}(\mathbf{I}+\boldsymbol{\alpha})\mathbf{B}_2\mathbf{f}'_0(z), & \text{in } S_1, \\ \mathbf{B}_2^{-1}(\mathbf{I}-i\boldsymbol{\beta})^{-1}(\boldsymbol{\alpha}+i\boldsymbol{\beta})\bar{\mathbf{B}}_2\bar{\mathbf{f}}'_0(z) + \mathbf{f}'_0(z), & \text{in } S_2, \end{cases} \tag{3.7}$$

where S_1, the upper half-space, and S_2, the lower half-space, are occupied by material 1 and 2, respectively. Here the subscripts 1 and 2 denote the quantities corresponding to material 1 and 2, respectively. Even if the material 1 is rigid or non-existent, the solution still remains valid. For the former case, $\boldsymbol{\alpha} = \mathbf{I}$ and $\boldsymbol{\beta} = \mathbf{S}_2\mathbf{W}_2^{-1}$, and therefore,

$$\mathbf{f}'(z) = \mathbf{f}'_0(z) - \mathbf{A}_2^{-1}\bar{\mathbf{A}}_2\bar{\mathbf{f}}'_0(z), \quad \text{in } S_2, \tag{3.8}$$

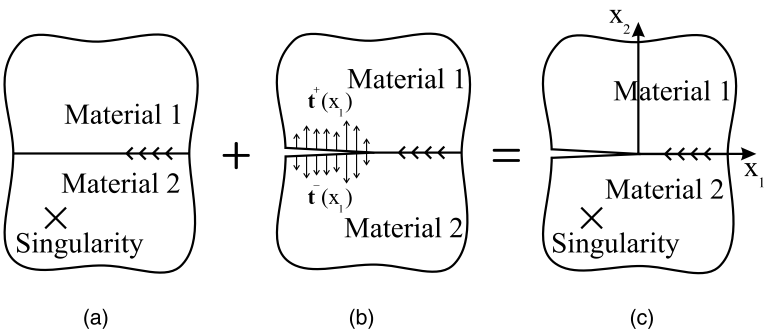

Fig. 1. Singularity and interfacial crack in dissimilar media.

while for the latter case, $\boldsymbol{\alpha} = -\mathbf{I}$, and therefore,

$$\mathbf{f}'(z) = \mathbf{f}'_0(z) - \mathbf{B}_2^{-1}\bar{\mathbf{B}}_2\bar{\mathbf{f}}'_0(z), \quad \text{in } S_2. \tag{3.9}$$

If the anisotropic/anisotropic bimaterial shown in Fig. 1(a) becomes an isotropic/isotropic one, by applying the equivalence theorem, the bimaterial solution, Eq. (3.7) with isotropic **A**, **B** inserted, reduces to

$$\mathbf{f}'(z) = [\Phi(z), \Omega(z), \omega(z)]^{\mathrm{T}} = \begin{cases} \left[\dfrac{1+\alpha}{1-\beta}\Phi_0(z), \dfrac{1+\alpha}{1+\beta}\Omega_0(z), (1-\gamma)\omega_0(z)\right]^{\mathrm{T}}, & \text{in } S_1, \\ \left[\Phi_0(z) + \dfrac{\alpha-\beta}{1+\beta}\bar{\Omega}_0(z), \Omega_0(z) + \dfrac{\alpha+\beta}{1-\beta}\bar{\Phi}_0(z), \omega_0(z) + \gamma\bar{\omega}_0(z)\right]^{\mathrm{T}}, & \text{in } S_2. \end{cases} \tag{3.10}$$

Here, $\mathbf{f}'_0(z)$ as given in Eq. (3.5) is used. This result coincides with that of Suo (1989) and Choi and Earmme (2002b). Also, for the case of the material 1 which is rigid or non-existent, by using Eq. (3.8) or (3.9), respectively, similar result is obtained, but is not shown here.

Now we suppose that material 1 shown in Fig. 1(a) is isotropic, while material 2 has cubic symmetry, in which the cubic axes coincide with the coordinate axes. The equivalence theorem also provide us with the solution of a singularity in an isotropic/cubic bimaterial from Eq. (3.7), which is explicitly written as

$$\mathbf{f}'(z) = \begin{cases} \begin{bmatrix} \Phi_1(z) \\ \Omega_1(z) \\ \omega_1(z) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2\pi}\dfrac{1+\alpha}{1+\beta}\left[(1+\mu_1\mathrm{i})\dfrac{q_1}{z-s_1} + (1+\mu_2\mathrm{i})\dfrac{q_2}{z-s_2}\right] \\ \dfrac{1}{2\pi}\dfrac{1+\alpha}{1-\beta}\left[(1-\mu_1\mathrm{i})\dfrac{q_1}{z-s_1} + (1-\mu_2\mathrm{i})\dfrac{q_2}{z-s_2}\right] \\ -\dfrac{(1+\gamma)q_3}{\pi(z-s_3)} \end{bmatrix}, & \text{in } S_1, \\ \mathbf{B}_2^{-1}(\mathbf{I}-\mathrm{i}\boldsymbol{\beta})^{-1}(\boldsymbol{\alpha}+\mathrm{i}\boldsymbol{\beta})\bar{\mathbf{B}}_2\bar{\mathbf{f}}'_0(z) + \mathbf{f}'_0(z), & \text{in } S_2. \end{cases} \tag{3.11}$$

In the upper half-space S_1, $z = x_1 + \mathrm{i}x_2$, while in the lower half-space S_2, z is reinterpreted by $z_j = x_1 + p_j x_2$, where p_j's are given by Eqs. (A.3a) and (A.3b) for the cubic material 2. In other words, the isotropic elasticity described in Section 2.2 should be used in the upper half-space, while the anisotropic elasticity given in Section 2.1 in the lower half-space is used. It is worth noting that inplane and antiplane deformations are generally coupled for isotropic/anisotropic bimaterials as they do for anisotropic/anisotropic bimaterials. However, the inplane and antiplane deformations given by Eq. (3.11) are decoupled, since the interface is {1 0 0} plane and the cubic axes coincide with the coordinate axes. The bimaterial solution, Eq. (3.7), is also applicable for cubic (material 1)/isotropic (material 2) bimaterial with the application of the equivalence theorem, however it is not shown here.

3.2. Near-tip field and stress intensity factor for interfacial crack

Consider a crack lying along the interface between two anisotropic materials with material 1 above and material 2 below as shown in Fig. 1(c). Crack tip lies on the plane $x_2 = 0$ at $x_1 = 0$. Near-tip fields are given by (Beom and Atluri, 1995)

$$\mathbf{f}'(z) = \begin{cases} \dfrac{1}{2\sqrt{2\pi z}}\mathbf{B}_1^{-1}(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})\mathbf{Y}[z^{-\mathrm{i}\varepsilon}]\mathbf{G}(z) + \mathbf{B}_1^{-1}(\mathbf{I}+\boldsymbol{\alpha})\mathbf{H}(z), & \text{in } S_1, \\ \dfrac{1}{2\sqrt{2\pi z}}\mathbf{B}_2^{-1}(\mathbf{I}-\mathrm{i}\boldsymbol{\beta})\mathbf{Y}[z^{-\mathrm{i}\varepsilon}]\mathbf{G}(z) + \mathbf{B}_2^{-1}(\mathbf{I}-\boldsymbol{\alpha})\mathbf{H}(z), & \text{in } S_2. \end{cases} \tag{3.12}$$

Here subscripts 1 and 2 refer to the materials 1 and 2, respectively, and $\mathbf{Y}[\zeta(z)]$ is explicitly defined in terms of the real bimaterial matrix $\boldsymbol{\beta}$ as (Qu and Li, 1991)

$$\mathbf{Y}[\zeta(z)] \equiv \mathbf{I} + \frac{\mathrm{i}}{2\beta}[\zeta(z) - \bar{\zeta}(z)]\boldsymbol{\beta} + \frac{1}{\beta^2}\left\{1 - \frac{1}{2}\left[\zeta(z) + \bar{\zeta}(z)\right]\right\}\boldsymbol{\beta}^2, \tag{3.13}$$

where $\zeta(z)$ is an arbitrary function of z. The matrix function $\mathbf{Y}[\zeta(z)]$ plays an important role in representing the oscillatory fields near the crack tip. A Williams type expansion of the near-tip field is generated from Eq. (3.12) by writing $\mathbf{G}(z)$ and $\mathbf{H}(z)$ in terms of Taylor series expansions as:

$$\mathbf{G}(z) = \sum_{n=0}^{\infty} \mathbf{a}_n z^n, \quad \mathbf{H}(z) = \sum_{n=0}^{\infty} \mathrm{i}\mathbf{b}_n z^n, \tag{3.14}$$

where $\mathbf{a}_n$ and $\mathbf{b}_n$ are real vectors. Then $\mathbf{a}_0$ represents the strength of the crack tip singularity, which was defined as the stress intensity factor by Qu and Li (1991), while the coefficient $\mathbf{b}_0$ represents a stress acting parallel to the crack surface (i.e., σ_{11}), which is referred to as the T-stress in the case of homogeneous material (Rice, 1988). The stress σ_{11} induced by $\mathbf{b}_0$ is uniform but different in each of the two materials.

The singular stress field along the bonded interface near the crack tip is given by

$$\{\sigma_{2j}(x_1)\} = \frac{1}{\sqrt{2\pi x_1}} \mathbf{Y}(x_1^{-\mathrm{i}\varepsilon})\mathbf{G}(x_1). \tag{3.15}$$

Thus, the vector of stress intensity factors which uniquely characterizes the singular field can be defined by (Qu and Li, 1991)

$$\mathbf{k} = \lim_{x_1 \to 0^+} \sqrt{2\pi x_1}\mathbf{Y}(x_1^{\mathrm{i}\varepsilon})\{\sigma_{2j}(x_1)\}, \tag{3.16}$$

where $\mathbf{k} = [K_2, K_1, K_3]^{\mathrm{T}}$. The analytic functions generating the singular part of the interface stress can be expressed in terms of $\mathbf{k}$ as

$$\mathbf{f}'(z) = \begin{cases} \dfrac{1}{2\sqrt{2\pi z}} \mathbf{B}_1^{-1}(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{Y}[z^{-\mathrm{i}\varepsilon}]\mathbf{k}, & \text{in } S_1, \\ \dfrac{1}{2\sqrt{2\pi z}} \mathbf{B}_2^{-1}(\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\mathbf{Y}[z^{-\mathrm{i}\varepsilon}]\mathbf{k}, & \text{in } S_2. \end{cases} \tag{3.17}$$

For isotropic/isotropic bimaterial case, by applying the equivalence theorem again, it is straightforward to show that Eq. (3.17) reduces to

$$\mathbf{f}'(z) = \begin{bmatrix} \Phi(z) \\ \Omega(z) \\ \omega(z) \end{bmatrix} = \begin{cases} \left[\dfrac{\bar{K}\mathrm{e}^{-\pi\varepsilon}z^{-1/2-\mathrm{i}\varepsilon}}{2\sqrt{2\pi}\cosh\pi\varepsilon}, \dfrac{K\mathrm{e}^{\pi\varepsilon}z^{-1/2+\mathrm{i}\varepsilon}}{2\sqrt{2\pi}\cosh\pi\varepsilon}, \dfrac{K_3}{2\sqrt{2\pi z}}\right]^{\mathrm{T}}, & \text{in } S_1, \\ \left[\dfrac{\bar{K}\mathrm{e}^{\pi\varepsilon}z^{-1/2-\mathrm{i}\varepsilon}}{2\sqrt{2\pi}\cosh\pi\varepsilon}, \dfrac{K\mathrm{e}^{-\pi\varepsilon}z^{-1/2+\mathrm{i}\varepsilon}}{2\sqrt{2\pi}\cosh\pi\varepsilon}, \dfrac{K_3}{2\sqrt{2\pi z}}\right]^{\mathrm{T}}, & \text{in } S_2. \end{cases} \tag{3.18}$$

The asymptotic fields near the interfacial crack tip in an isotropic/cubic bimaterial (the material 1 is isotropic, while the material 2 has cubic symmetry) are also obtained from Eq. (3.17), and explicitly expressed as

$$\mathbf{f}'(z) = \begin{cases} \begin{bmatrix} \Phi_1(z) \\ \Omega_1(z) \\ \omega_1(z) \end{bmatrix} = \left[\dfrac{\bar{K}\mathrm{e}^{-\pi\varepsilon}z^{-1/2-\mathrm{i}\varepsilon}}{2\sqrt{2\pi}\cosh\pi\varepsilon}, \dfrac{K\mathrm{e}^{\pi\varepsilon}z^{-1/2+\mathrm{i}\varepsilon}}{2\sqrt{2\pi}\cosh\pi\varepsilon}, \dfrac{K_3}{2\sqrt{2\pi z}}\right]^{\mathrm{T}}, & \text{in } S_1, \\ \dfrac{1}{2\sqrt{2\pi z}} \mathbf{B}_2^{-1}(\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\mathbf{Y}[z^{-\mathrm{i}\varepsilon}]\mathbf{k}, & \text{in } S_2. \end{cases} \tag{3.19}$$

Note that the complex potentials given in S_1 of Eq. (3.19) are identical to those for the isotropic/isotropic bimaterial given in S_1 of Eq. (3.18). Therefore, the anisotropy of the lower material contributes to the asymptotic field in S_1 only through the oscillatory index ε (or equivalently β).

Although $\mathbf{k}$ defined in Eq. (3.16) does not have the proper dimension (Qu and Li, 1991), it provides a unique characterization of the crack tip state. A stress intensity factor with the dimensions of stress times square root of length, denoted by $\hat{\mathbf{k}}_l$ can also be defined based on the characteristic length l as suggested by Rice (1988) for the isotropic case, i.e., $\hat{\mathbf{k}}_l$ is related to $\mathbf{k}$ by $\hat{\mathbf{k}}_l = \mathbf{Y}(l^{-i\varepsilon})\mathbf{k}$. It is noted that the stress intensity factor $\mathbf{k}$ given in Eq. (3.16) for the oscillatory field recovers the classical stress intensity factor $[K_{\mathrm{II}}, K_I, K_{\mathrm{III}}]^{\mathrm{T}}$ as the bimaterial continuum reduces to a homogeneous one. In the isotropic case, the definition of the stress intensity factor given in Eq. (3.16) also reduces to that for the case of an isotropic/isotropic interface crack. It is also obvious that the stress intensity factor defined in Eq. (3.16) uniquely characterizes the interfacial crack tip state even in an anisotropic/isotropic bimaterial and is compatible with the definition of the conventional stress intensity factor for isotropic/isotropic and anisotropic/anisotropic interface cracks.

3.3. Interaction between singularity and interfacial crack

Consider a semi-infinite interfacial crack in an anisotropic/anisotropic bimaterial interacting with a singularity as shown in Fig. 2(a), which can be solved by the superposition scheme illustrated in Fig. 1. Since we have the solution, Eq. (3.7), for Fig. 1(a), the problem in Fig. 1(b) is only discussed here, in which the tractions $\mathbf{t}^+(x_1) = -\mathbf{t}^-(x_1) = -\mathbf{t}(x_1)$ are applied on the upper and lower surfaces of the crack, respectively. The stress intensity factors are given by (Beom and Atluri, 1996)

$$\mathbf{k} = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{0} \mathbf{Y}\left[(-x_1)^{i\varepsilon}\cosh\pi\varepsilon\right]\mathbf{t}(x_1)\frac{\mathrm{d}x_1}{\sqrt{-x_1}}. \tag{3.20}$$

For the special case in which $\varepsilon = 0$ (non-oscillatory field), the matrix function $\mathbf{Y}$ in Eq. (3.20) is simply replaced by $\mathbf{I}$, which yields the identical result to that of Suo (1990). For the general isotropic/isotropic bimaterial case, we utilize the equivalence theorem; hence, Eq. (3.20) with $\mathbf{Y}$ (Eq. (3.13)) evaluated for the isotropic/isotropic bimaterial reduces to

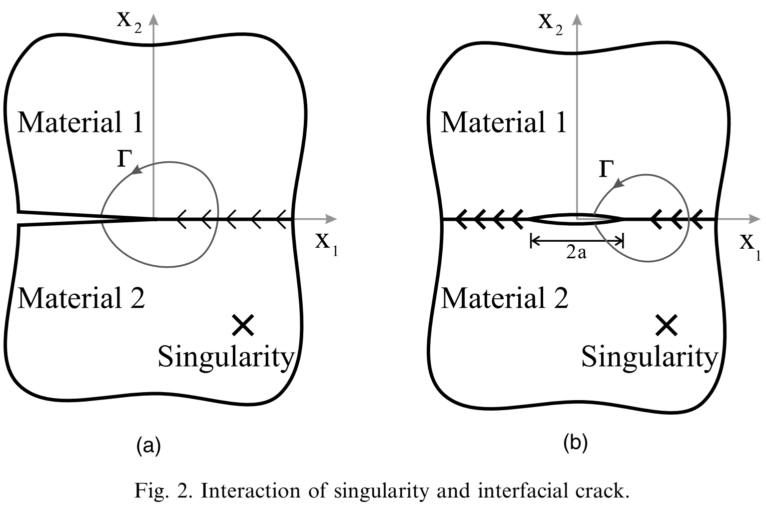

Fig. 2. Interaction of singularity and interfacial crack.

$$\mathbf{k} = \begin{bmatrix} K_2 \\ K_1 \\ K_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{\pi}}\cosh \pi\varepsilon \,\mathrm{Im}\left\{ \int_{-\infty}^{0} [t_2(x_1) + \mathrm{i}t_1(x_1)](-x_1)^{-1/2-\mathrm{i}\varepsilon}\,\mathrm{d}x_1 \right\} \\ \sqrt{\frac{2}{\pi}}\cosh \pi\varepsilon \,\mathrm{Re}\left\{ \int_{-\infty}^{0} [t_2(x_1) + \mathrm{i}t_1(x_1)](-x_1)^{-1/2-\mathrm{i}\varepsilon}\,\mathrm{d}x_1 \right\} \\ \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \frac{t_3(x_1)\,\mathrm{d}x_1}{\sqrt{-x_1}} \end{bmatrix}, \tag{3.21}$$

which coincides with the result of Suo (1989) and Cho et al. (1994). The stress intensity factors for a semi-infinite crack in an anisotropic/anisotropic bimaterial interacting with singularities such as a point force and a dislocation as shown in Fig. 1(c) can be calculated from Eq. (3.20) with the tractions from Eq. (3.7) given as

$$\mathbf{t}(x_1) = (\mathbf{I} + \mathrm{i}\boldsymbol{\beta})^{-1}(\mathbf{I} + \boldsymbol{\alpha})\mathbf{B}_2\mathbf{f}_0'(x_1). \tag{3.22}$$

Next, a finite interfacial crack in the interval $(-a, a)$ in an anisotropic/anisotropic bimaterial interacting with a singularity as shown in Fig. 2(b) is considered. The stress intensity factors for the finite crack are obtained by (Beom and Atluri, 1996)

$$\mathbf{k} = \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} \mathbf{Y}\left[\left(2a\frac{a - x_1}{a + x_1}\right)^{\mathrm{i}\varepsilon} \cosh \pi\varepsilon\right]\mathbf{t}(x_1)\sqrt{\frac{a + x_1}{a - x_1}}\mathrm{d}x_1, \tag{3.23}$$

where the tractions $\mathbf{t}(x_1)$ are given by Eq. (3.22). For the isotropic/isotropic bimaterial case, Eq. (3.23) reduces to

$$\mathbf{k} = \begin{bmatrix} K_2 \\ K_1 \\ K_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{\pi}}\cosh \pi\varepsilon \,\mathrm{Im}\left\{ \int_{-a}^{a} [t_2(x_1) + \mathrm{i}t_1(x_1)]\left(2a\frac{a - x_1}{a + x_1}\right)^{-1/2-i\varepsilon} \mathrm{d}x_1 \right\} \\ \sqrt{\frac{2}{\pi}}\cosh \pi\varepsilon \,\mathrm{Re}\left\{ \int_{-a}^{a} [t_2(x_1) + \mathrm{i}t_1(x_1)]\left(2a\frac{a - x_1}{a + x_1}\right)^{-1/2-\mathrm{i}\varepsilon} \mathrm{d}x_1 \right\} \\ \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} t_3(x_1)\left(\frac{a + x_1}{a - x_1}\right)^{1/2} \mathrm{d}x_1 \end{bmatrix}. \tag{3.24}$$

which coincides with the result of Suo (1989). It is emphasized again that the methods given in this section are valid not only for anisotropic (homogeneous), isotropic (homogeneous), anisotropic/anisotropic, and isotropic/isotropic materials but also for anisotropic/isotropic material, with the aid of the equivalence theorem.

4. Conservation integral

The equivalence theorem proposed in Section 2 was utilized in the previous section for the problems of singularities and interfacial cracks in dissimilar media. The theorem also provides us with a clue to the similarity of conservation integrals in anisotropic solids and in isotropic solids, which is explored in this section.

4.1. Path-independent property and complex variable form

The well-known J integral for an elastic solid, which is homogeneous in the x_1 direction, is defined by (Rice, 1968)

$$J\{\mathbf{u};\Gamma\} = \int_\Gamma \left(\mathscr{W} n_1 - \mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial x_1}\right)\mathrm{d}s. \tag{4.1}$$

Here $\mathscr{W}$ is the strain energy density, $\mathbf{n}$ is the unit outward normal vector, $\mathbf{t}$ is the surface traction, Γ is a path connecting any two points on the opposite sides of the crack surface and enclosing the crack tip and ds is an element of arc length along Γ as shown in Fig. 3. It is well known that the J integral is independent of any path Γ, and has the physical meaning of energy release rate due to crack extension. Yeh et al. (1993) obtained the complex form of the J integral for an anisotropic solid. They used a different normalization of $\mathbf{A}$ and $\mathbf{B}$ from ours, and Kim et al. (2001) recently showed that the J integral can be written for the arbitrarily normalized $\mathbf{A}$ and $\mathbf{B}$ matrices as

$$J\{\mathbf{u};\Gamma\} = 2\mathrm{Im}\left[\int_\Gamma \mathbf{f}'^{\mathrm{T}}(z)\mathbf{B}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{f}'(z)\,\mathrm{d}z\right]. \tag{4.2}$$

It is noted that z should be properly replaced by z_1, z_2, and z_3 before the manipulation of vectors and matrices, that is, the J integral is written in a component form as

$$J\{\mathbf{u};\Gamma\} = 2\mathrm{Im}\left[\sum_{j,k,m=1}^{3}\int_\Gamma f_j'(z_j)B_{kj}L_{km}^{-1}B_{mj}f_j'(z_j)\,\mathrm{d}z_j\right]. \tag{4.3}$$

For isotropic materials, Budiansky and Rice (1973) obtained the J integral in terms of Muskhelishvili complex potentials, which is slightly modified in the following form:

$$\begin{aligned} J\{\mathbf{u};\Gamma\} &= 2\mathrm{Im}\left\{\int_{z_1^0}^{z_2^0}\left[\frac{\kappa+1}{4G}\Phi(z)\Omega(z)+\frac{1}{4G}\omega(z)\omega(z)\right]\mathrm{d}z - \left[\frac{\kappa+1}{4G}x_2\Phi(z)\Phi(z)\right]_{z_1^0}^{z_2^0}\right\} \\ &= 2\mathrm{Im}\left\{\int_\Gamma \mathbf{f}'^{\mathrm{T}}(z)\mathbf{B}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\mathbf{f}'(z)\,\mathrm{d}z - \left[\frac{\kappa+1}{4G}x_2\Phi(z)\Phi(z)\right]_{z_1^0}^{z_2^0}\right\}, \end{aligned} \tag{4.4}$$

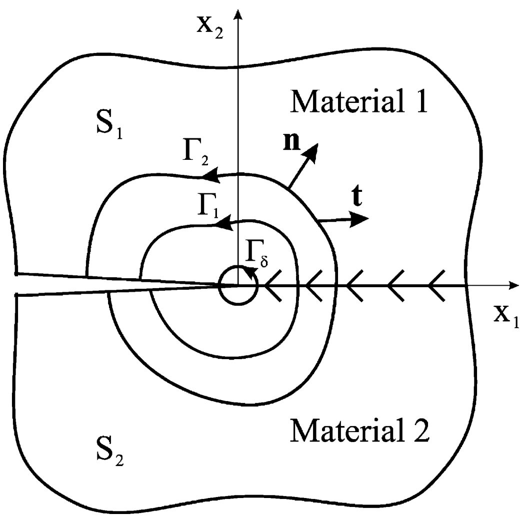

Fig. 3. Paths of conservation integrals.

in which z_1^0 and z_2^0 correspond to the end points of the contour Γ and $\mathbf{f}'(z)$, $\mathbf{B}$, and $\mathbf{L}$ are those for isotropic solids. It is remarked here that the J integral in a complex potential form for isotropic solids is identical to that for anisotropic solids, provided that both end points of Γ are on x_1-axis ($x_2 = 0$). Therefore, the J integral in Eq. (4.2) is regarded as a general form expressed in terms of complex potentials for both anisotropic and isotropic solids. In other words, the J integral in Eq. (4.2) may be used to obtain the energy release rate due to crack extension in isotropic (homogeneous), anisotropic (homogeneous), isotropic/isotropic, anisotropic/anisotropic, or isotropic/anisotropic materials, provided that the complex potentials are known. The relation between the J integral and stress intensity factors can be derived through the complex formula of the J integral in Eq. (4.2) with near-tip fields of an interfacial crack given by Eq. (3.17), resulting in (Beom and Atluri, 1996)

$$J\{\mathbf{u};\Gamma\} = \tfrac{1}{4}\mathbf{k}^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{k}. \tag{4.5}$$

Here

$$\mathbf{U}^{-1} = (\mathbf{L}_1^{-1} + \mathbf{L}_2^{-1})(\mathbf{I} + \boldsymbol{\beta}^2). \tag{4.6}$$

Even if one or both of the constituent materials of the bimaterial are isotropic, the relation (4.5) with Eq. (4.6) still remains valid. For example, for isotropic/isotropic bimaterials the J integral in Eq. (4.5) reduces to

$$J\{\mathbf{u};\Gamma\} = \frac{1}{16\cosh^2\pi\varepsilon}\left(\frac{\kappa_1+1}{G_1} + \frac{\kappa_2+1}{G_2}\right)(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2) + \frac{1}{4}\left(\frac{1}{G_1} + \frac{1}{G_2}\right)K_{\mathrm{III}}^2. \tag{4.7}$$

Here, Eqs. (2.19) and (2.24) are used for $\mathbf{L}^{-1} = \mathrm{Re}\{\mathbf{M}^{-1}\}$ and $\boldsymbol{\beta}$, respectively, in the evaluation of Eq. (4.6).

In order to know the individual component of $\mathbf{k}$, the mutual integral proposed by Chen and Shield (1977) is found to be useful and it was exploited by Choi and Earmme (1992) even for circular arc-shaped interfacial crack. Consider two independent equilibrium states of an elastically deformed bimaterial body, with each displacement being denoted by $\mathbf{u}$ and $\tilde{\mathbf{u}}$, respectively. The J-based mutual integral for the two states, denoted by $\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma\}$, is defined by (Chen and Shield, 1977)

$$\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma\} = \int_\Gamma \left[tr(\tilde{\boldsymbol{\sigma}}\cdot\nabla\mathbf{u})n_1 - \mathbf{t}\cdot\frac{\partial\tilde{\mathbf{u}}}{\partial x_1} - \tilde{\mathbf{t}}\cdot\frac{\partial\mathbf{u}}{\partial x_1}\right]\mathrm{d}s, \tag{4.8}$$

where overscript tilde ($\sim$) represents the quantities associated with the equilibrium state $\tilde{\mathbf{u}}$. As noted by Chen and Shield (1977), $\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma\}$ can be written in terms of the J integral as

$$\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma\} = J\{\mathbf{u}+\tilde{\mathbf{u}};\Gamma\} - J\{\mathbf{u};\Gamma\} - J\{\tilde{\mathbf{u}};\Gamma\}. \tag{4.9}$$

The $\mathscr{M}$ integral satisfies the same conservation law as that of the J integral. Thus we have the following conservation law:

$$\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma_0\} = 0. \tag{4.10}$$

Here an area A enclosed by Γ_0 containing the interface bonded perfectly is assumed to be free from any singularities. This conservation law has been applied to the direct calculation of stress intensity factors without actually solving complicated boundary value problems for isotropic bimaterial (Cho et al., 1994) as well as for anisotropic bimaterial (Beom and Atluri, 1996). Making use of the complex form of the J integral and the relation between J integral and $\mathscr{M}$ integral in Eq. (4.9), it can be shown that the complex form of the $\mathscr{M}$ integral is given by

$$\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma\} = 4\mathrm{Im}\left[\int_\Gamma \mathbf{f}'^{\mathrm{T}}(z)\mathbf{B}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\tilde{\mathbf{f}}'(z)\,\mathrm{d}z\right] \tag{4.11}$$

for anisotropic solids and

$$\mathscr{M}\{\mathbf{u},\tilde{\mathbf{u}};\Gamma\} = 4\mathrm{Im}\left\{ \int_\Gamma \mathbf{f}'^{\mathrm{T}}(z)\mathbf{B}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{B}\tilde{\mathbf{f}}'(z)\,\mathrm{d}z - \left[\frac{\kappa+1}{4G}x_2\Phi(z)\tilde{\Phi}(z)\right]_{z_1}^{z_2}\right\} \tag{4.12}$$

for isotropic solids. It is noted that $\mathscr{M}$ integral also has the same complex form for anisotropic and isotropic solids, provided that both end points of Γ are on x_1-axis. Therefore, the mutual integral $\mathscr{M}$ in Eq. (4.11) may also be used in isotropic (homogeneous), anisotropic (homogeneous), isotropic/isotropic, anisotropic/anisotropic, or isotropic/anisotropic materials, provided that the complex potentials are known.

4.2. Auxiliary field and $\mathscr{M}$ integral

The mutual integral $\mathscr{M}$ defined by Eq. (4.8) is useful to determine the individual stress intensity factors K_1, K_2 and K_3 as well as T-stresses for the equilibrium state $\mathbf{u}$ as suggested by Chen and Shield (1977). Though they dealt with the crack in a homogeneous isotropic medium, the mutual integral $\mathscr{M}$ can be used to analyze the interfacial crack in anisotropic/anisotropic, isotropic/isotropic, and anisotropic/isotropic bimaterials, provided that the solution for another equilibrium state $\tilde{\mathbf{u}}$, called the auxiliary solution, is known. The auxiliary fields should satisfy (i) the equilibrium equation in the domain enclosed by the contour Γ and (ii) the traction-free boundary condition on the crack surface enclosed by Γ. Due to the equivalence theorem, the auxiliary solution for anisotropic/anisotropic bimaterial can also be used to obtain the auxiliary solution for anisotropic (homogeneous), isotropic (homogeneous), isotropic/isotropic, and anisotropic/isotropic materials. Now we summarize the auxiliary fields for the semi-infinite crack (Fig. 2(a)) and the finite crack (Fig. 2(b)) as follows (Beom and Atluri, 1996; Kim et al., 2001):

$$\tilde{\mathbf{f}}'^{(j)}(z) = \begin{cases} \dfrac{1}{2\sqrt{2\pi z}}\mathbf{B}_1^{-1}(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})\mathbf{Y}(z^{-\mathrm{i}\varepsilon})\hat{\mathbf{e}}^j, & \text{in } S_1,\\ \dfrac{1}{2\sqrt{2\pi z}}\mathbf{B}_2^{-1}(\mathbf{I}-\mathrm{i}\boldsymbol{\beta})\mathbf{Y}(z^{-\mathrm{i}\varepsilon})\hat{\mathbf{e}}^j, & \text{in } S_2, \end{cases} \tag{4.13}$$

$$\tilde{\mathbf{f}}'^{(j)}(z) = \begin{cases} \dfrac{1}{4\sqrt{\pi a}}\sqrt{\dfrac{z+a}{z-a}}\mathbf{B}_1^{-1}(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})\mathbf{Y}\left[\left(2a\dfrac{z-a}{z+a}\right)^{-\mathrm{i}\varepsilon}\right]\hat{\mathbf{e}}^j, & \text{in } S_1,\\ \dfrac{1}{4\sqrt{\pi a}}\sqrt{\dfrac{z+a}{z-a}}\mathbf{B}_2^{-1}(\mathbf{I}-\mathrm{i}\boldsymbol{\beta})\mathbf{Y}\left[\left(2a\dfrac{z-a}{z+a}\right)^{-\mathrm{i}\varepsilon}\right]\hat{\mathbf{e}}^j, & \text{in } S_2, \end{cases} \tag{4.14}$$

$$\tilde{\mathbf{f}}'^{(j)}(z) = \begin{cases} \dfrac{1}{2\pi\mathrm{i}z}\mathbf{B}_1^{-1}(\mathbf{I}+\boldsymbol{\alpha})\hat{\mathbf{e}}^j, & \text{in } S_1,\\ \dfrac{1}{2\pi\mathrm{i}z}\mathbf{B}_2^{-1}(\mathbf{I}-\boldsymbol{\alpha})\hat{\mathbf{e}}^j, & \text{in } S_2, \end{cases} \tag{4.15}$$

$$\tilde{\mathbf{f}}'^{(j)}(z) = \begin{cases} \dfrac{1}{2\pi\mathrm{i}(z-a)}\mathbf{B}_1^{-1}(\mathbf{I}+\boldsymbol{\alpha})\hat{\mathbf{e}}^j, & \text{in } S_1,\\ \dfrac{1}{2\pi\mathrm{i}(z-a)}\mathbf{B}_2^{-1}(\mathbf{I}-\boldsymbol{\alpha})\hat{\mathbf{e}}^j, & \text{in } S_2. \end{cases} \tag{4.16}$$

Here $\hat{\mathbf{e}}^j$ ($j = 1,2,3$) is the base vector with the component $\hat{\mathbf{e}}^j_m = \delta_{jm}$, where δ_{jm} is the Kronecker delta. Eqs. (4.13) and (4.14) may be used to determine the individual stress intensity factors K_1, K_2 and K_3 for the semi-infinite crack and the finite crack, respectively, while T-stresses can be determined by using Eqs. (4.15) and (4.16) for the semi-infinite crack and the finite crack, respectively.

Now, we introduce the conservation integral $\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma\}$, where $\tilde{\mathbf{u}}^{(j)}$ are the displacements generated by the complex potentials given by one of Eqs. (4.13)–(4.16). The $\mathscr{M}$ integrals, which are evaluated with $\mathbf{u}$ and the auxiliary fields, are related to stress intensity factors and T-stresses by

$$k_m = 2\sum_{j=1}^{3} U_{mj}\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma\}, \quad (m = 1, 2, 3) \tag{4.17a}$$

$$T_m = \frac{1}{8}\sum_{j=1}^{3} V_{mj}\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma\}, \quad (m = 1, 2, 3) \tag{4.17b}$$

where $\mathbf{U}$ is given in Eq. (4.6), and $\mathbf{V} = \mathbf{L}_1 + \mathbf{L}_2$. It is remarked that $\mathbf{k} = [k_1, k_2, k_3]^{\mathrm{T}} = [K_2, K_1, K_3]^{\mathrm{T}}$. Here $\mathbf{T} = \{T_m\}$ corresponds to $\mathbf{b}_0$ in Williams type expansions (i.e., Eq. (3.12) with Eq. (3.14)) representing stress acting parallel to the crack surface. It is obvious that the mutual integrals $\mathscr{M}$ in Eqs. (4.17a) and (4.17b) provide a sufficient information for determination of the individual stress intensity factors K_1, K_2 and K_3 as well as T-stresses σ_{11}^0 and σ_{31}^0. The $\mathscr{M}$ integral has the same path-independence as that of the J integral, therefore, Eqs. (4.17a) and (4.17b) are valid for any path Γ tracing from the lower crack face to the upper crack face. It is worth mentioning that the mutual integrals $\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma\}$ can be exploited to calculate individual stress intensity factors and T-stresses numerically for a finite body. In the analysis of crack problem by means of computational method, such as the finite element method, a fundamental difficulty is encountered in efforts to compute the values of field quantities near the crack tip. The mutual integrals can be evaluated along the contour remote from the crack tip where the numerical fields are more accurate.

4.3. Application of $\mathscr{M}$ integral to interfacial crack-singularity interaction

Two crack configurations in an infinite bimaterial as shown in Fig. 2(a) and (b) are considered, in which the singularities such as a point load and a dislocation are embedded in the bimaterial. The interaction problem can be solved by the superposition scheme as described in Section 3.3. However, to obtain the stress intensity factors and T-stresses of the crack-singularity interaction problems, it is more convenient to apply the conservation laws $\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma_0\} = 0$ without actually solving the boundary value problem. In this section, based on the equivalence theorem and the analogy of the conservation integrals described in the previous subsections, we utilize the mutual integral $\mathscr{M}$ to solve the problem of interfacial crack-singularity interaction, and therefore the results given in this section are valid for anisotropic (homogeneous), isotropic (homogeneous), anisotropic/anisotropic, isotropic/isotropic, and anisotropic/isotropic materials.

First, let us consider the semi-infinite crack as shown in Fig. 2(a). A contour Γ_0 consisting of $\Gamma_c^+ + \Gamma_c^- + \Gamma^+ + \Gamma^- + \Gamma_\infty - \Gamma_s - \Gamma_\delta$ as shown in Fig. 4(a) is chosen to compute the right hand sides of Eq. (4.17a). Here Γ^+ and Γ^- are the interior paths, and Γ_s is a vanishingly small path enclosing the point $z = s$. The line integral over the parts $\Gamma^+ + \Gamma^-$ makes no contribution to $\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma_0\}$. Furthermore, there is no contribution from the infinitely large circle Γ_∞ because the stress components are $\sigma_{ij} = \mathrm{O}(1/r)$ and also $\tilde{\sigma}_{ij} = \mathrm{O}(1/r)$ as $r \to \infty$. Thus, the conservation law $\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma_0\}$ implies

$$\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma_\delta\} = -\mathscr{M}\{\mathbf{u}, \tilde{\mathbf{u}}^{(j)}; \Gamma_s\}. \tag{4.18}$$

Potentials near the singularity at $z = s$ can be written as (Suo, 1990)

$$\mathbf{f}'(z) = \frac{\mathbf{q}}{2\pi(z - s)} + \frac{(\bar{s} - s)\mathbf{Q}\cdot\mathbf{q}}{2\pi(z - s)^2} + \hat{\mathbf{f}}'(z), \tag{4.19}$$

where $\hat{\mathbf{f}}'(z)$ is analytic at $z = s$ (but has a branch cut along the crack surface), and $Q_{ij} = 0$ for an anisotropic material and $Q_{ij} = \delta_{i2}\delta_{j1}$ for an isotropic material as described in Section 3.1. Eq. (4.19) is regarded as a

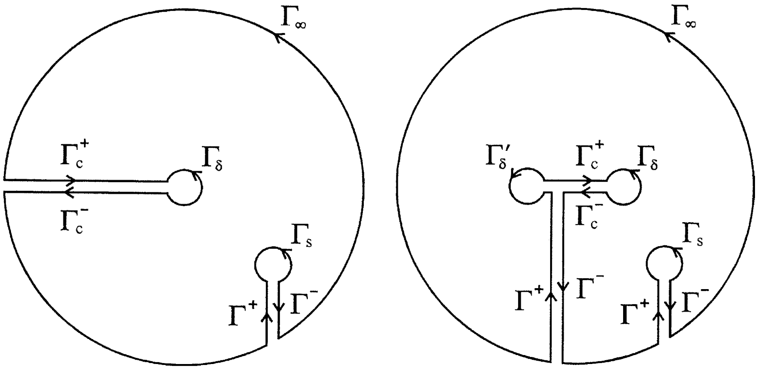

Fig. 4. Integration contours for (a) the semi-infinite crack and (b) the finite crack.

general form of potentials near a point singularity regardless of whether anisotropic or isotropic materials, and it provides us with the unified approach for both anisotropic and isotropic materials at the expense of the mathematical simplicity. Substituting Eqs. (4.19) and (4.13) into Eq. (4.11), the integral in Eq. (4.18) corresponding to each auxiliary field is easily evaluated by its residues and then substituted into Eq. (4.17a), resulting in

$$k_m = \sqrt{\frac{2}{\pi}} \sum_{j,k,n,r=1}^{3} \mathrm{Re}\left\{ -2Y_{mn}(s_j^{\mathrm{i}\varepsilon})X_{nr}(\mathbf{L}_2^{-1})_{rk}(\mathbf{B}_2)_{kj}\frac{q_j}{\sqrt{s_j}} + Y_{mn}[(1-2\varepsilon i)s_j^{\mathrm{i}\varepsilon}]X_{nr}(\mathbf{L}_2^{-1})_{rk}(\mathbf{B}_2)_{kj}\frac{(\mathbf{Q}_2\cdot\mathbf{q})_j}{\sqrt{s_j}}\left(\frac{\bar{s}_j}{s_j}-1\right)\right\}. \tag{4.20}$$

Here $\mathbf{X} = (\mathbf{I}-\mathrm{i}\boldsymbol{\beta})^{-1}(\mathbf{L}_1^{-1}+\mathbf{L}_2^{-1})^{-1}$, and $k_1 = K_2$, $k_2 = K_1$, and $k_3 = K_3$. By using the auxiliary field, Eq. (4.15), instead of Eq. (4.13) and following the same procedure that was used to obtain Eq. (4.20), we can get

$$T_m = -\mathrm{Im}\left\{ \sum_{j,k=1}^{3} \frac{1}{2\pi s_j}(\mathbf{L}_2\mathbf{B}_2^{-\mathrm{T}})_{mj}(\mathbf{B}_2^{\mathrm{T}}\mathbf{L}_2^{-1})_{jk}(\mathbf{B}_2)_{kj}\left[q_j - \left(\frac{\bar{s}_j}{s_j}-1\right)(\mathbf{Q}_2\cdot\mathbf{q})_j\right]\right\}, \tag{4.21}$$

for the semi-infinite crack of Fig. 2(a).

For the finite crack as shown in Fig. 2(b), by applying the similar arguments used for the semi-infinite crack, in which the contour Γ_0 consists of $\Gamma_c^+ + \Gamma_c^- + \Gamma^+ + \Gamma^- + \Gamma_\infty - \Gamma_s - \Gamma_\delta - \Gamma_\delta'$ as shown in Fig. 4(b) and the auxiliary fields, Eqs. (4.14) and (4.16), are used instead of Eqs. (4.13) and (4.15), respectively, the stress intensity factors and T-stresses for the finite crack are obtained as

$$k_m = -\frac{2}{\sqrt{\pi a}} \sum_{j,k,n,r,s=1}^{3} U_{mk}\mathrm{Re}\left\{ Y_{nk}\left[\left(2a\frac{s_j-a}{s_j+a}\right)^{-\mathrm{i}\varepsilon}\right](\mathbf{I}-\mathrm{i}\boldsymbol{\beta})_{rn}(\mathbf{L}_2^{-1})_{rs}(\mathbf{B}_2)_{sj}q_j\sqrt{\frac{s_j+a}{s_j-a}} - Y_{nk}\left[(1+2\varepsilon i)\left(2a\frac{s_j-a}{s_j+a}\right)^{-\mathrm{i}\varepsilon}\right](\mathbf{I}-\mathrm{i}\boldsymbol{\beta})_{rn}(\mathbf{L}_2^{-1})_{rs}(\mathbf{B}_2)_{sj}\frac{(\mathbf{Q}_2\cdot\mathbf{q})_j(\bar{s}_j-s_j)a}{\sqrt{s_j^2-a^2}(s_j-a)}\right\} + \frac{1}{\sqrt{\pi a}}\sum_{j,k=1}^{3} U_{mj}Y_{kj}[(2a)^{-\mathrm{i}\varepsilon}](\mathbf{b}+\boldsymbol{\beta}^{\mathrm{T}}\mathbf{L}_1^{-1}\mathbf{p}+\mathbf{W}_1\mathbf{p})_k, \tag{4.22}$$

$$T_m = -\mathrm{Im}\left\{ \sum_{j,k=1}^{3} \frac{1}{2\pi(s_j - a)} (\mathbf{L}_2\mathbf{B}_2^{-T})_{mj} (\mathbf{B}_2^{\mathrm{T}}\mathbf{L}_2^{-1})_{jk} (\mathbf{B}_2)_{kj} \left[q_j - \frac{\bar{s}_j - s_j}{s_j - a} (\mathbf{Q}_2 \cdot \mathbf{q})_j \right] \right\}. \quad (4.23)$$

When material 2 is anisotropic, $\mathbf{Q}_2$ becomes zero matrix and therefore the stress intensity factors and T-stresses given in Eqs. (4.20)–(4.23) are quite simplified.

5. Summary

The equivalence between the anisotropic and isotropic elasticity for two-dimensional deformation under certain conditions is demonstrated in this paper. The isotropic elasticity can be reconstructed in the same framework of the anisotropic elasticity, when the interface between dissimilar media lies along a straight line. Therefore, many known solutions for an anisotropic bimaterial are directly used for a bimaterial, of which one or both of the constituent materials are isotropic. The equivalence is useful to obtain the solutions for singularities and cracks in an anisotropic/isotropic bimaterial without solving the boundary value problems directly. The interaction solutions of singularities, interfaces, and cracks in dissimilar anisotropic media are summarized, to be used for the cases of isotropic/isotropic and anisotropic/isotropic bimaterials. Conservation integrals also have the similar analogy between the anisotropic and isotropic elasticity so that J integral and J -based mutual integral $\mathscr{M}$ are expressed in the same complex forms for the anisotropic and isotropic materials, when both end points of the integration paths are on the straight interface. The use of J and $\mathscr{M}$ integrals together with the present equivalence are exemplified to obtain energy release rate, stress intensity factors, and T-stresses of interfacial cracks lying in the interface of anisotropic/anisotropic, isotropic/isotropic, and anisotropic/isotropic bimaterials.

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Appendix A. Elastic matrices for cubic materials

The characteristic equation (2.6) for a cubic material, of which the cubic axes coincide with the coordinate axes, reduces to

$$p^4 + \eta p^2 + 1 = 0, \quad \text{for inplane deformation,} \quad (A.1a)$$

$$p^2 + 1 = 0, \quad \text{for antiplane deformation,} \quad (A.1b)$$

in which η is expressed in terms of three independent elastic constants C_{11}, C_{12}, and C_{66} as

$$\eta = \frac{C_{11}^2 - C_{12}^2 - 2C_{12}C_{66}}{C_{11}C_{66}}, \quad (A.2)$$

The roots of the characteristic equations (A.1), which cannot be real (Lekhnitskii, 1963), are

$$\begin{cases} p_1 = \frac{1}{2}(\mathrm{i}\sqrt{\eta+2} + \sqrt{2-\eta}), & p_2 = \frac{1}{2}(\mathrm{i}\sqrt{\eta+2} - \sqrt{2-\eta}), & \text{for } -2 < \eta < 2, \\ p_1 = \frac{1}{2}(\sqrt{\eta+2} + \sqrt{\eta-2}), & p_2 = \frac{1}{2}(\sqrt{\eta+2} - \sqrt{\eta-2}), & \text{for } 2 < \eta, \end{cases} \quad (A.3a)$$

$$p_3 = \mathrm{i}. \quad (A.3b)$$

Here the roots with positive imaginary parts are chosen. In the above eigenvalues, the degenerate case, $\eta = 2$, corresponds to transversely isotropic materials, which is not shown here. The anisotropic matrices defined in Eqs. (2.6)–(2.8) reduce to

$$\mathbf{A} = \frac{1}{C_{66}} \begin{bmatrix} -\dfrac{p_1^2 C_{11} + C_{66}}{p_1^2 C_{11} - C_{12}} & -\dfrac{p_2^2 C_{11} + C_{66}}{p_2^2 C_{11} - C_{12}} & 0 \\ \dfrac{(C_{12} + C_{66}) p_1}{p_1^2 C_{11} - C_{12}} & \dfrac{(C_{12} + C_{66}) p_2}{p_2^2 C_{11} - C_{12}} & 0 \\ 0 & 0 & \mathrm{i} \end{bmatrix}, \tag{A.4}$$

$$\mathbf{B} = \begin{bmatrix} -p_1 & -p_2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{A.5}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \dfrac{C_{11}\sqrt{\eta + 2}}{C_{11}^2 - C_{12}^2} & \dfrac{\mathrm{i}}{C_{11} + C_{12}} & 0 \\ -\dfrac{\mathrm{i}}{C_{11} + C_{12}} & \dfrac{C_{11}\sqrt{\eta + 2}}{C_{11}^2 - C_{12}^2} & 0 \\ 0 & 0 & \dfrac{1}{C_{66}} \end{bmatrix}, \tag{A.6}$$

in which an arbitrary normalizing factor for **A** and **B** is chosen so that the matrix **B** is non-dimensional, and the matrix **A** has the dimension of compliance, and therefore the matrix **M** has the dimension of stiffness.

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